Identifying Codes over L-graphs

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Castell de Cardona 15 September 2011
Outline of the Talk

1. Introduction and Motivation
2. Introduction to L-graphs
3. Identifying codes over $\mathbb{Z}^2$ and L-graphs
4. Conclusions and Future Work
Identifying codes in multiprocessor systems

Multiprocessor systems are modeled by a graph of nodes with the edges representing the communications.

Multiprocessor systems can have faulty nodes. Hence we need fault diagnosis.

Fault diagnosis can be approached by identifying codes.
Definition of Identifying codes

- Let $G = (V, E)$ be a graph. $G$ induces a metric $d$, counting the number of edges in shortest paths.

- The **ball** of radius $r$ centered in $v \in V$ is $B_r(v) = \{ w \in V | d(v,w) \leq r \}$. We say that $v$ $r$-**covers** the elements in its ball.

- Given a code $C \subseteq V$, we define $K_r(v) = C \cap B_r(v)$. When $K_r(v) \neq K_r(w)$ we say that $v$ and $w$ are $r$-**separated**.

**Definition ([KarChaLev TIT98])**

$C$ is called $r$-**identifying** when the sets $K_r(v)$ are nonempty and different.
Identification on grids

Identifying codes have been considered over several infinite grids:

We construct a wide family of identifying codes over the square grid. Previous work on the square grid: [HonLob JCT02], [Charon et al. EJC01] and [Charon et al EJC02].
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L-graphs

- In applications like multiprocessor systems finite graphs are required.
- We consider the graph obtained by $\mathbb{Z}^2$ modulo a lattice and call it an L-graph.

$\Rightarrow$ modulo a lattice
Outline of the Talk

1. Introduction and Motivation

2. Introduction to L-graphs
   - Cayley graphs
   - Multidimensional Circulant: L-graphs
   - Examples

3. Identifying codes over $\mathbb{Z}^2$ and L-graphs

4. Conclusions and Future Work
Definition of Cayley Graphs

**Definition**

Given a group $\Gamma$ and an adjacency set $A \subset \Gamma$, the Cayley graph $G(\Gamma; A)$ is defined as the graph with vertices $V = \Gamma$ and edges

$$E = \{(v, v + g) | g \in A\}$$

We assume:

1. $A = -A$ (undirected)
2. $0 \notin A$ (simple)
3. $\langle A \rangle = \Gamma$ (connected)

The infinite square grid can be defined as a Cayley Graph over $\Gamma = \mathbb{Z}^2$ and $A = \left\{ \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. 
Multidimensional Circulant: L-graphs

**Congruences in** $\mathbb{Z}^n$

Let $M \in \mathcal{M}_{n \times n}(\mathbb{Z})$ an $n \times n$ integer **matrix**.

Two **vectors** $a, b \in \mathbb{Z}^n$ are congruent modulo $M$:

$$a \equiv b \pmod{M} \iff a - b \in M\mathbb{Z}^n,$$

where

$$M\mathbb{Z}^n = \{c_1 m_1 + \ldots + c_n m_n : c_i \in \mathbb{Z}\}.$$

We denote as $\mathbb{Z}^n/M\mathbb{Z}^n$, the **group** of integer vectors module $M$, which has $|\det(M)|$ elements when $M$ is nonsingular.
Definition of L-graphs

**Definition ([Fiol DM95])**

We call *multidimensional circulant* to the Cayley graph with group $V = \Gamma = \mathbb{Z}^n/M\mathbb{Z}^n$ and adjacencies $A \subseteq \mathbb{Z}^n$.

For $n = 2$ we call them **L-graphs**.

We use adjacencies $A = \{\pm e_1, \pm e_2\} = \{\pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$. But this is an assumption which *doesn’t lose generality*.

They will be denoted as $L(M)$. 
Example: Products of cycles

\[ M = \begin{pmatrix} 10 & 0 \\ 0 & 6 \end{pmatrix} \]

Cartesian
\[ A = \{ (\pm 1, 0)^t, (0, \pm 1)^t \} \]

Kronecker
\[ A = \{ \pm (1, 1)^t, \pm (1, -1)^t \} \]
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   - Our construction over $\mathbb{Z}^2$
   - Identifying over L-graphs

4. Conclusions and Future Work
**R-fixed balls**

**Definition**

Let $C$ be a code of $\mathbb{Z}^2$. The ball $B_r(v)$ is **$r$-fixed** by $C$ if there exists at least a codeword in each one of the next sets:

$$\mathcal{UR}_r^-(v) \cup \mathcal{BL}_r^+(v), \mathcal{UL}_r^-(v) \cup \mathcal{BR}_r^+(v), \mathcal{BR}_r^-(v) \cup \mathcal{UL}_r^+(v), \mathcal{BL}_r^-(v) \cup \mathcal{UR}_r^+(v)$$
Our construction over $\mathbb{Z}^2$

**Lemma**

For all $v \in \mathbb{Z}^2$, $B_r(v)$ is $r$-fixed by $C = \langle \left( \frac{t}{t+d} \right), \left( -\frac{(t+d)}{t} \right) \rangle$, where $r = t^2 + dt + \frac{d^2 - 1}{2}$ and $\gcd(2t, d) = 1$.

**Remark**

In general a code with $r$-fixed balls is not $r$-identifying.
Lemma

For all \( v \in \mathbb{Z}^2 \), \( B_r(v) \) is \( r \)-fixed by \( C = \langle \left( \begin{array}{c} t \\ t+d \end{array} \right), \left( -\left( \begin{array}{c} t+d \\ t \end{array} \right) \right) \rangle \), where \( r = t^2 + dt + \frac{d^2-1}{2} \) and \( \gcd(2t, d) = 1. \)

Remark

In general a code with \( r \)-fixed balls is not \( r \)-identifying.
Lemma

For all $v \in \mathbb{Z}^2$, $B_r(v)$ is $r$-fixed by $C = \langle \left( \begin{array}{c} t \\ t+d \end{array} \right), \left( \begin{array}{c} -zt \\ -zt \end{array} \right) \rangle$, where $r = t^2 + dt + \frac{d^2-1}{2}$ and $\gcd(2t, d) = 1$.

Remark

In general a code with $r$-fixed balls is not $r$-identifying.
Lemma

For all $v \in \mathbb{Z}^2$, $B_r(v)$ is $r$-fixed by $C = \langle \left( \begin{array}{c} t \\ t + d \end{array} \right), \left( -\frac{(t+d)}{t} \right) \rangle$, where $r = t^2 + dt + \frac{d^2 - 1}{2}$ and $\gcd(2t, d) = 1$.

Remark

In general a code with $r$-fixed balls is not $r$-identifying.
Sketch of the proof

Let $v$ be a vertex. We prove that $B_r(v)$ is $r$-fixed, we look for a codeword $c = v + \left( \frac{x}{y} \right) \in \mathcal{U} \mathcal{R}_r^-(v) \cup \mathcal{B} \mathcal{L}_r^+(v)$.

1. When $\gcd(2t, d) = 1$ we find $x_0, y_0$ with $x_0 + y_0 = r$.
2. Other solutions $x' = x_0 + \lambda N$, $y' = y_0 + \lambda N$ with $N = 2r + 1 = t^2 + (t + d)^2$.
3. By Euclidean division we take $\lambda$ such that $0 \leq x' < N$.
   - If $0 \leq x' \leq r$ then $c = v + \left( \frac{x'}{y'} \right)$ is in $\mathcal{U} \mathcal{R}_r^-(v)$.
   - If $r < x' \leq 2r$ so $-r \leq y' < 0$. Then $c = v + \left( \frac{x'}{y'-N} \right)$ is a codeword in $\mathcal{B} \mathcal{L}_r^+(v)$. 

\[ \text{Sketch of a } \mathbb{Z}^2 \text{ lattice with a point marked.} \]
Our construction over $\mathbb{Z}^2$

**Main result of the paper**

**Theorem**

Let $t, d$ be positive integers such that $\gcd(2t, d) = 1$, then

$$C = \langle \begin{pmatrix} t \\ t + d \end{pmatrix}, \begin{pmatrix} -(t + d) \\ t \end{pmatrix} \rangle$$

is an $r$-identifying code over $\mathbb{Z}^2$ for $r = t^2 + dt + \frac{d^2 - 1}{2}$. 

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**Sketch of the proof**

We want to see that $v \in V$ is $r$-separated from $w = v + (x, y)$.

1. We can assume $x > y \geq 0$.
2. Since balls are fixed, we have $c_1 \in U\mathcal{L}_r^-(v) \cup B\mathcal{R}_r^+(v)$
   - a. If $c_1 \in U\mathcal{L}_r^-(v)$ ✓
   - b. If $c_1 = v + \left(\frac{a}{r-1}\right) \in K_r(w)$ ✓
   - c. If $a > \frac{d-1}{2}$, we construct a codeword in $K_r(v)$ ✓
   - d. Else, $1 \leq a \leq \frac{d-1}{2}$ must be hold.
3. For $B\mathcal{L}_r^-(v) \cup U\mathcal{R}_r^+(v)$ we obtain $1 \leq b \leq t$.
4. Note that 2d and 3 cannot be hold simultaneously.
Identifying over L-graphs

An identifying code over $\mathbb{Z}^2$ which is periodic will be identifying for some L-graph. We need congruency (keeping density) and an lower bound in its size.

**Corollary**

Let $T = \begin{pmatrix} t & -(t+d) \\ t+d & t \end{pmatrix}$ and $M$ such that $\exists Q \in M_{2 \times 2}(\mathbb{Z})$, $M = TQ$ and $L(M)$ has a systole of length $> 2r + 1$. Then,

$$C = \langle \begin{pmatrix} t \\ t+d \end{pmatrix}, \begin{pmatrix} -(t+d) \\ t \end{pmatrix} \rangle$$

is an $r$-identifying code over $L(M)$ for $r = t^2 + dt + \frac{d^2-1}{2}$. 
Perfect Codes

Corollary

Let \( T = \begin{pmatrix} t & -(t+1) \\ t+1 & t \end{pmatrix} \) and \( M \) such that exists \( Q \in M_{2 \times 2}(\mathbb{Z}) \) satisfying \( M = TQ \) and that \( L(M) \) has a \textit{systole} of length \( \min\{d(0, Mg) | 0 \neq g \in \mathbb{Z}^2\} > 2r + 1 \). Then,

\[
C = \langle \begin{pmatrix} t \\ t+1 \end{pmatrix}, \begin{pmatrix} -(t+1) \\ t \end{pmatrix} \rangle
\]

is both a \( t(t+1) \)-\textit{identifying} code and \( t \)-\textit{perfect error correcting} code over \( L(M) \).
Density of the construction

The density of our construction is:

\[
D = \lim_{n \to \infty} \frac{|C \cap Q_n|}{|Q_n|} = \frac{|C|}{|V|} = \frac{\det(M)}{\det(T)} = \frac{1}{\det(T)} = \frac{1}{2r + 1}.
\]

Which is worse than the current best construction of \( \frac{2}{5r} \) [HonLob JCT02].

However, our codes have a small covering radius and include all the perfect codes over \( \mathbb{Z}^2 \) and L-graphs. This is useful at the first stage of some applications.
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Conclusions and Future Work

- We have presented lattice-based codes:
  - Wide family of identifying codes.
  - Small covering, our family includes the perfect codes.
  - Novel strategy: $r$-fixed balls.
- Could this codes work in other meshes?:
  - triangular $\square$,
  - hexagonal $\sqcap\sqcup$,
  - king $\mathbb{F}$. 
Thanks!

Questions?
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Apéndice

5. Introducción a L-grafos

6. Códigos perfectos

7. Identification
Enlaces Ortonormales

$$\det(A) \times G(M; A) \simeq G(\text{adj}(A)M; \det(A)I) \simeq m \times G(M')$$

$$C_{12}(3, 4) \simeq G\left(\begin{pmatrix} 12 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}\right) \simeq G\left(\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}\right)$$
El producto de Kronecker de dos ciclos admite una expresión directa como \textit{L-grafo} con enlaces ortonormales:

\textbf{Theorem ([MarCamBei ISIT10])}

\( C_a \times C_b \) es isomorfo a:

- \( a \geq b \) impar \( \Rightarrow \) \( L(M) \) donde \( M = \begin{pmatrix} \frac{a+b}{2} & \frac{a-b}{2} \\ \frac{a-b}{2} & \frac{a+b}{2} \end{pmatrix} \).

- \( a \geq b \) par \( \Rightarrow \) Dos copias disjuntas de \( L(M) \), donde

  \[
  M = \begin{pmatrix} \frac{a}{2} & -\frac{b}{2} \\ 0 & \frac{b}{2} \end{pmatrix}.
  \]

- \( a \) par, \( b \) impar \( \Rightarrow \) \( L(M) \), donde \( M = \begin{pmatrix} \frac{a}{2} & -b \\ 0 & b \end{pmatrix} \).

Para el resto de L-grafos tenemos un \textbf{algoritmo} que deja los enlaces ortonormales.
L-forma

El L-grafo asociado a $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ puede verse como:

\begin{align*}
b \leq 0 \leq a, c, d & \rightarrow \text{suma de rectángulos. } L(\begin{pmatrix} 6 & -5 \\ 4 & 7 \end{pmatrix}). \\
0 \leq a, b, c, d & \rightarrow \text{diferencia de rectángulos. } L(\begin{pmatrix} 10 & 6 \\ 4 & 8 \end{pmatrix}).
\end{align*}
El L-grafo asociado a $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ puede verse como:

$b \leq 0 \leq a, c, d \rightarrow$ suma de rectángulos. $L(\begin{pmatrix} 6 & -5 \\ 4 & 7 \end{pmatrix})$. 
Códigos Perfectos

We denote the ball of radius $t$ centered in a vertex $v$ of $L(M; E)$ as $B_M(v, t) = \{w \in \mathbb{Z}^2/M\mathbb{Z}^2 \mid d_M(v, w) \leq t\}$. Clearly, $B_M(v, t)$ has $2t^2 + 2t + 1$ elements.

**Definition**

A perfect $t$-error correcting code over an L-graph $L(M; A)$ is a subset $C$ of the vertex set such that the set of vertices is a disjoint union of balls as follows:

$$\mathbb{Z}^2/M\mathbb{Z}^2 = \bigcup_{c \in C} B_M(c, t).$$

The previous definition is equivalent to:

**Lemma**

$C \subset \mathbb{Z}^2/M\mathbb{Z}^2$ is a perfect $t$-error correcting code over $L(M; A)$ if and only if it is a subset of $\frac{|\det(M)|}{2t^2 + 2t + 1}$ vertices with the property $d_M(c, c') \geq 2t + 1$, for all $c, c' \in C$ such that $c \neq c'$. 
Códigos sobre Gaussianos

El retículo de los enteros Gaussianos ha sido usado para modelar constelaciones tipo QAM en \([\ldots]\).

La distancia de los grafos Gaussianos nos da una métrica y códigos correctores de errores sobre ellas \([\ldots]\).
Códigos sobre Gaussianos

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Códigos sobre Gaussianos

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Código en el grafo Gaussiano sobre $\mathbb{Z}[i]_{3+4i}$.

La distancia de los grafos Gaussianos nos da una métrica y códigos correctores de errores sobre ellas [?].
Theorem

Let \( M \in \mathcal{M}_{2 \times 2}(\mathbb{Z}) \) be a non singular matrix such that there exist \( Q \in \mathcal{M}_{2 \times 2}(\mathbb{Z}) \) with \( M = TQ \) (respectively \( M = T^t Q \)). Then, \( T + M\mathbb{Z}^2 \) is a perfect \( t \)-error correcting code over the graph \( L(M; E) \) (respectively \( T^t + \mathbb{Z}^2/M\mathbb{Z}^2 \)).

Proof. Let us prove the case in which \( M = TQ \). Since the equality of cardinalities is obtained by Corollary ??, next we prove that given any pair of different words of the code there is a distance between them greater than \( 2t + 1 \). In this aim, let \( T\gamma, T\gamma' \) be such that \( T\gamma \neq T\gamma' \) (mod \( M \)). Since \( d_M(T\gamma, T\gamma') = d_M(T(\gamma - \gamma'), 0) \), it is enough to show that \( d_M(T\gamma, 0) \geq 2t + 1 \) for any \( \gamma \in \mathbb{Z}^2 \) such that \( T\gamma \neq 0 \) (mod \( M \)).
If $d_M(T\gamma, 0) < 2t + 1$ we have that $T\gamma \equiv \begin{pmatrix} x \\ y \end{pmatrix} \pmod{M}$ such that $|x| + |y| < 2t + 1$ is minimum. Hence,

$$T\gamma - \begin{pmatrix} x \\ y \end{pmatrix} = M\gamma' = TQ\gamma',$$

which implies $T\gamma - TQ\gamma' = \begin{pmatrix} x \\ y \end{pmatrix}$, that is,

$$\begin{pmatrix} x \\ y \end{pmatrix} = T(I_2\gamma - Q\gamma') = T\gamma'',$$

with $|x| + |y| < 2t + 1$, which is not possible by Lemma ??.
**Remark**

Let $M$ such that $M \neq TQ$ for any $Q \in M_{2 \times 2}(\mathbb{Z})$. Therefore, it is straightforward that $T + \mathbb{Z}^2/M\mathbb{Z}^2 = \mathbb{Z}^2/M\mathbb{Z}^2$, which implies that it is not a perfect $t$-error correcting code. Clearly, any perfect code containing another vertex $c$ different to zero is a translation of the one containing vertex zero: $c + T \pmod{M}$.

As a direct consequence of the previous results and remark we have that:

**Corollary**

$L(M;E)$ has a perfect $t$-error correcting code if and only if there exists $Q \in M_{2 \times 2}(\mathbb{Z})$ such that $M = TQ$ (or $M = T^tQ$).
Idea de la identificación

Nos interesa la mínima densidad $\rightarrow$ Montones de artículos sobre cotas de la densidad mínima.
Idea de la identificación

Nos interesa la mínima densidad $\Rightarrow$ Montones de artículos sobre cotas de la densidad mínima.
Let $t, d \geq 0$ be two integers such that $\gcd(2t, d) = 1$ and let us define $r = t^2 + dt + \frac{d^2-1}{2}$. We consider the code $\mathcal{C} = \langle \left(\begin{array}{c} t \\ t+d \end{array}\right), \left(\begin{array}{c} -(t+d) \\ t \end{array}\right) \rangle$ of the infinite mesh. We denote also by $\mathcal{C} = \left(\begin{array}{cc} t & -(t+d) \\ t+d & t \end{array}\right)$ the matrix associated to the code. If $N$ is the cardinal of $\mathbb{Z}^2/\mathcal{C}\mathbb{Z}^2$, we have that $N = t^2 + (t + d)^2 = 2r + 1$. Since

$$\left(\begin{array}{cc} t & -(t+d) \\ t+d & t \end{array}\right) \left(\begin{array}{c} t+d \\ t \end{array}\right) = \begin{pmatrix} 0 \\ N \end{pmatrix}$$

and

$$\left(\begin{array}{cc} t & -(t+d) \\ t+d & t \end{array}\right) \left(\begin{array}{c} t \\ -t-d \end{array}\right) = \begin{pmatrix} N \\ 0 \end{pmatrix}$$

we obtain that if $\left(\begin{array}{c} x \\ y \end{array}\right) \in \mathcal{C}$ then $\forall m, n \in \mathbb{Z}$, $\left(\begin{array}{c} x+mN \\ y+aN \end{array}\right) \in \mathcal{C}$. We will use this fact to prove the following:
For all $v \in V$, $B_r(v)$ is $r$-fixed by $C((t + d), -(t + d))$, where $r = t^2 + dt + \frac{d^2 - 1}{2}$ and $\gcd(2t, d) = 1$.

Proof. We will prove that for each $v \in V$ there exists a codeword $c$ in the set $\mathcal{U}R_r^-(v) \cup \mathcal{B}L_r^+(v)$. The other three codewords can be obtained by making the suitable rotations. Note that, if $c \in \mathcal{U}R_r^-(v) \cup \mathcal{B}L_r^+(v)$ and $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $Rc$ is codeword in $\mathcal{U}L_r^-(Rv) \cup \mathcal{B}L_r^+(Rv)$, $R^2c$ is codeword in $\mathcal{B}L_r^-(R^2v) \cup \mathcal{U}R_r^+(R^2v)$ and $R^3c$ is codeword in $\mathcal{B}R_r^-(R^3v) \cup \mathcal{U}L_r^+(R^3v)$. Since $c$ must be in $C$, there must exist integers $a, b$ such that $c = \left( \frac{at - b(t + d)}{a(t + d) + bt} \right)$. Now, if $v = \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right)$, let us denote $\left( \begin{array}{c} x \\ y \end{array} \right) = c - v$. Then, we obtain the following system of Diophantine equations:

\[
\begin{cases}
  x = at - b(t + d) - v_1 \\
  y = a(t + d) + bt - v_2
\end{cases}
\]
Therefore, \( x + y = a(2t + d) - bd - v_1 - v_2 \). By hypothesis, 
\( \gcd(2t + d, d) = \gcd(2t, d) = 1 \), which implies that there exist \( a_0, b_0 \in \mathbb{Z} \) such that \( a_0(2t + d) - b_0d = r + v_1 + v_2 \), thus there also exist \( x_0, y_0 \in \mathbb{Z} \) such that \( x_0 + y_0 = r \). The set of solutions is 
\[ \{ a' = a_0 + d\lambda, b' = b_0 + (2t + d)\lambda \mid \lambda \in \mathbb{Z} \}. \]
Hence, 
\[ x' = a_0t - b_0(t+d) + td\lambda - (t+d)(2t + d)\lambda = a_0t - b_0(t+d) - N\lambda = x_0 + \lambda N \]
and in the same way \( y' = y_0 + \lambda N \).
Now, we set \( \lambda \) such that \( 0 \leq x' < N = 2r + 1 \), so \( 0 \leq x' \leq 2r \). Now,

- If \( 0 \leq x' \leq r \) then we have that \( 0 \leq y' \leq r \). Therefore, taking 
  \( (x, y) = (x', y') \) we obtain that \( c = v + (x, y) \) is in \( \mathcal{U} \mathcal{R}_r^-(v) \).

- Otherwise, we have that \( r < x' \leq 2r \) so \( -r \leq y' < 0 \). Therefore, 
  taking \( (x, y) = (x', y'^N) \) we have that \( x + y = r - N = -r - 1 \) and 
  \( -r - 1 < x \leq -1 \), that is \( -r \leq x < 0 \), thus obtaining that 
  \( c = v + (x, y) \) is a codeword in \( \mathcal{B} \mathcal{L}_r^+(v) \).
Theorem

Let $t, d$ be positive integers such that $\gcd(2t, d) = 1$ and $r = t^2 + dt + \frac{d^2 - 1}{2}$. Then, $\mathcal{C} = \langle (\frac{t}{t+d}), (\frac{-t}{t+d}) \rangle$ is a $r$-identifying code in $\mathbb{Z}^2$.

Proof. By previous lemma we have that all $r$-balls are $r$-fixed. However, we know that this is not enough for being a $r$-identifying code. Therefore, we will next prove that any $v \in V$ is $r$-separated from $w = v + \left( \begin{array}{c} x \\ y \end{array} \right)$, for any $\left( \begin{array}{c} x \\ y \end{array} \right) \in \mathbb{Z}^2$. Note that, if $|x| = |y|$ this is straightforward since the balls are $r$-fixed. Then, by symmetry, we can assume without loss of generality that $x > y \geq 0$.
If there is a codeword in $\mathcal{U}\mathcal{L}_r^-(v)$ then it is straightforward that they are $r$-separated. Otherwise, we have a codeword $c_1 \in \mathcal{C} \cap \mathcal{BR}_r^+(v)$. 
which is not in $K_r(v)$. If $c_1$ is in $K_r(w)$ we have finished. As $c_1 \in \mathcal{BR}_r^+(v)$ it can be expressed as $c_1 = v + \binom{a}{a-r-1}$. Then

$$c_1 + \binom{-(t+d)}{t} = v + \binom{a-(t+d)}{a+t-r-1}$$

is a codeword which is further from $K_r(w)$ than $c_1$. And when $a > \frac{d-1}{2}$, it is in $K_r(v)$.

If we do the same than in previous case, but for $\mathcal{BL}_r^-(v) \cup \mathcal{UR}_r^+(v)$, we obtain $c_2 = v + \binom{b}{r-b+1}$. Then, we consider

$$c_2 + \binom{-(t+d)}{t} - \binom{t}{t+d} = c_2 + \binom{-2t-d}{-d} = v + \binom{b+t}{r-b+t+d+1}$$

which is further from $K_r(w)$ than $c_2$ and when $b > t$ it is in $K_r(v)$.

To finish we need to prove that the case when both codewords $c_1, c_2$ are in the left ranges never happen. For that, when $1 \leq a \leq \frac{d-1}{2}$ and $1 \leq b \leq t$, with $c_1, c_2 \in \mathcal{C}$ should be $c = c_1 - c_2 \in \mathcal{C}$ but we will obtain a
contradiction. We have that \( c = c_1 - c_2 = (a + b - 1 - N) = (x' \ y') \) is in the rectangle defined by:

\[
\{ 1 - t \leq x' < \frac{d - 1}{2} - 1 < d, \ 1 - N \leq y' \leq t + \frac{d - 1}{2} - 1 - N < t + d - N \}
\]

which does not include the point \((0 \ N)\). This rectangle is a subset of

\[
\{ 1 - 2t \leq x' \leq -t, \ N \leq y' \leq N + t - 1 \} \cup \\
\{ 1 - t \leq x' \leq d, \ N \leq y' \leq N + t + d - 1 \}
\]

which is a set of representatives of \( \mathbb{Z}^2 / C \mathbb{Z}^2 \) (see [JorPot MM65]) and contains \((0 \ N)\). As a consequence, all its elements are different. Therefore, \( c \) is not congruent with \((0 \ 0)\) and thus it is not a codeword, which is a contradiction.
Corollary

Let \( T = \begin{pmatrix} t & -(t + d) \\ t + d & t \end{pmatrix} \) and \( M \) be such that there exists \( Q \in \mathcal{M}_{2 \times 2}(\mathbb{Z}) \) such that \( M = TQ \) and \( L(M) \) have a systolic length \( \min\{d(0, M\gamma) \mid 0 \neq \gamma \in \mathbb{Z}^2\} > 2r + 1 \). Then, \( C = \langle (\frac{t}{t+d}) , (\frac{-(t+d)}{t}) \rangle \) is an \( r \)-identifying code in \( L(M) \) for \( r = t^2 + dt + \frac{d^2 - 1}{2} \).

Remark

Remember that the systolic length \( \min\{d(0, M\gamma) \mid 0 \neq \gamma \in \mathbb{Z}^2\} > 2r + 1 \) is the length of the shortest non-trivial cycle in \( L(M) \).